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(*) Corresponding author
Peer review method: Double-blind
Original research article
DOI: 10.20544/HORIZONS.1.1.24.P14
Section B



MODERN ASPECTS OF QUASICONTINUOUS FUNCTIONS

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Abstract

The concept of continuity became an inspiration for the creation of topology. In the 20th century, there were many attempts to generalize this concept. One such attempt was made by the Polish mathematician Stefan Kempisty in the 1920s, when he introduced the concept of quasicontinuity. Various concepts of continuity in topological spaces are defined and their interconnections are examined. In this paper, we consider only Kempisty's quasicontinuity for functions of several real variables, in order to give a new approach to these functions regarding various applications in many fields of technical sciences. New proofs are provided for three of his fundamental theorems from his original work. A new generalization of Kempisty's theorem is made with three new theorems, each of which is individually stronger than Kempisty's theorem and which address the issues posed by his theorem. Two of these theorems, in general form, were proved by J. Ewert, who obtained them while examining multiplied functions, but the proofs given in this paper do not utilize the mentioned results of J. Ewert.

Keywords: quasicontinuous function, symmetrically quasicontinuous function, upper quasicontinuous function, lower quasicontinuous function, equally quasicontinuous functions.

INTRODUCTION

The concept of continuity has played a significant role in the development of mathematical analysis, but also beyond it. In fact, it can be said that this concept became an inspiration for the creation of topology.

In the 20th century, there were many attempts to generalize this concept. One such attempt was by the Polish mathematician Stefan Kempisty in the 1920s, who introduced the concept of quasicontinuity. In (Seweryn-kuzmanovska and Chalamani 2014), various concepts of continuity in topological spaces are defined and their relationships are examined.

The purpose of this paper is to give a new insight of Kempisty's quasicontinuity in light of modern topological terminology. We use his original work (Kempisty 1932), but we have replaced many of his proofs with our own. This applies to the proofs of Theorems 2.1, 4.1, and 4.2. Additionally, one of theorems shown by Kempisty in (Kempisty 1932) has been replaced with three Theorems: 5.1, 5.2, and 5.3. Each of these theorems is individually stronger than Kempisty's theorem and addresses the issues posed by his theorem. As we know, two of these Theorems, namely 5.2 and 5.3, were proved in a general form in the works (Ewert 1982) and (Ewert 1984) by J. Ewert, obtained while Ewert was examining multifunctions. However, the proofs provided in this paper do not utilize the aforementioned results of J. Ewert.

QUASICONTINUOUS FUNCTIONS

Let $\mathbb{R}$ be a set of real numbers and let $\mathbb{R}^n$ is n -dimensional Cartesian space.

Definition 2.1. In space $\mathbb{R}^n$, the set of form:

$$P = (a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n) = \prod_{i=1}^n (a_i, b_i)$$

is called *open n -dimensional interval*.

All intervals considered here will be non-degenerate, i.e. $a_i < b_i$ for $i = 1, 2, \dots, n$.

Definition 2.2. A neighborhood V_x of the point $x = (x_1, x_2, \dots, x_n)$ is any interval of form $(x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r) \times \dots \times (x_n - r, x_n + r)$.

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be an arbitrary function. The notion of quasicontinuity, introduced earlier in (Seweryn-Kuzmanovska and Chalamani 2014) (Definition 2.1), here we define as follows:

Definition 2.3. A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is called *quasicontinuous* at the point $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$, if for an arbitrary $\varepsilon > 0$, there exists an interval in every neighborhood of that point, in which the inequality

$$|f(x) - f(x^0)| < \varepsilon$$

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ continuous from one side at some point, is quasicontinuous in that point.

A function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ continuous at a point in the direction of some angle (with vertex located at the same point), is also quasicontinuous.

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ that is quasicontinuous everywhere is obviously pointwise discontinuous, but there are pointwise discontinuous functions that are not quasicontinuous. An example of such a function is the characteristic function of a finite set of points.

Note 2.1. If in every interval there is a point of quasicontinuity, then the function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is also pointwise discontinuous.

Proof. Let $U \subset \mathbb{R}^n$ be a non-empty open set. We will show that within the set U there exists a point of continuity of the function f .

Let P_1 be a closed interval that is contained in U . According to the assumption, there exists the point $x_1 \in \text{Int } P_1$ which is a point of quasicontinuity of a function f . Then, there exists a closed interval $P_2 \subset \text{Int } P_1$ such that for each point $x \in P_2$ the inequality holds:

$$|f(x) - f(x_1)| < 1.$$

Let $x_2 \in \text{Int } P_2$ be the accumulation point of the function f . Then there exists a closed interval $P_3 \subset \text{Int } P_2$ such that for each point $x \in P_3$ the condition is satisfied

$$|f(x) - f(x_1)| < \frac{1}{2}.$$

Proceeding in the same way, in the k -th step of mathematical induction we find a point $x_k \in \text{Int } P_k$ and a closed interval $P_{k+1} \subset \text{Int } P_k$ such that for each point $x \in P_{k+1}$ holds:

$$|f(x) - f(x_1)| < \frac{1}{k}.$$

Without losing generality, it can be assumed that $d(P_k) < \frac{1}{k}$.

Let x_0 be a common point for all intervals P_k ($k=1,2,\dots$). It is easy to see that the function f is continuous at the point x_0 . ♦

We will prove the following

Theorem 2.1. If a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is quasicontinuous with respect to every variable, then f is quasicontinuous.

Proof. First, let suppose that $n=2$. We consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. Let $(x_0, y_0) \in \mathbb{R}^2$ is a fixed point and $\varepsilon > 0$. Let $V_{(x_0, y_0)} = I \times J$ is a neighborhood of the point (x_0, y_0) where I, J are open intervals. Since the function f is quasicontinuous with respect to x , there is an open interval $I_1 \subset I$ such that

$$|f(x, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2}, \text{ for } x \in I_1.$$

Since all functions $f_x(t) = f(x, t)$ are quasicontinuous, then for each $x \in I_1$ there exists an open interval $J(x) \subset J$ with endpoints rational numbers such that

$$|f(x, y_0) - f(x, y)| < \frac{\varepsilon}{2}, \text{ for } y \in J(x).$$

The set of open intervals with rational endpoints is countable. Then there is an interval $J_1 \subset J$ such that the set $\{x \in I_1 \mid J(x) = J_1\}$ is of the second category. Hence, there is an open interval $I_2 \subset I_1$ such that the set $A = \{x \in I_2 \mid J(x) = J_1\}$ is dense in I_2 . Let $U = I_2 \times J_1$ and $(x_1, y_1) \in U$. Then,

$$|f(x_1, y_1) - f(x_0, y_0)| < \varepsilon.$$

Hence, it follows the quasicontinuity of the function f at the point (x_0, y_0) . Indeed, otherwise we would have

$$|f(x_1, y_1) - f(x_0, y_0)| \geq \varepsilon \quad (1)$$

and according to the quasicontinuity of the function $f^{y_1}(t) = f(t, y_1)$ in point x_1 would exist an open interval $I_3 \subset I_2$, such that

$$|f(x, y_1) - f(x_1, y_1)| \geq \frac{\varepsilon}{2} \text{ for } x \in I_3.$$

However, from the density of the set A in I_2 it follows that there is a point $x_3 \in A \cap I_3$ and therefore

$$|f(x_3, y_1) - f(x_0, y_0)| \leq |f(x_3, y_1) - f(x_3, y_0)| + |f(x_3, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is the opposite of (1).

For a function $f: \square^n \rightarrow \square$, that is quasicontinuous with respect to each of the variables $x_1, x_2, \dots, x_n$, we can apply the above procedure for the variables x_1 and x_2 , then for the point (x_1, x_2) and the variable, x_3 , and finally we can complete the proof by applying this procedure to the point $(x_1, x_2, \dots, x_{n-1})$ and the variables x_n .

Since every quasicontinuous function is point discontinuous, it follows that a function that is quasicontinuous with respect to each of the variables is point discontinuous. ♦

Corollary 2.1. Theorem shown by (Hahn 1921). A quasicontinuous function with respect to each of the variables is point discontinuous.

SYMMETRIC QUASICONTINUOUS FUNCTIONS

We say that n -dimensional open interval

$$P = (a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n)$$

is intersected by the hyperplane $x_i = c$ ($i = 1, 2, \dots, n$) if $a_i < c < b_i$.

Definition 3.1. A function $f: \square^n \rightarrow \square$ is said to be *symmetrically quasincontinuous* with respect to x_i ($i = 1, 2, \dots, n$) at a point $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ if in every open neighborhood of that point there is an interval intersected by a hyperplane $x_i = x_i^0$ in which

$$|f(x) - f(x^0)| < \varepsilon.$$

It is obvious that the function $f: \square^n \rightarrow \square$ is symmetrically quasicontinuous with respect to x_n and is quasicontinuous with respect to $(x_1, x_2, \dots, x_n)$.

A set that is defined by the conditions:

$$\begin{aligned} a_i < x_i < b_i \quad \text{for } i = 1, 2, \dots, n-1, \\ x_n &= c \end{aligned}$$

is an interval on a hyperplane $x_n = c$.

If $f: \square^n \rightarrow \square$ is symmetrically quasicontinuous with respect to x_n , then each interval of the hyperplane $x_n = c$ contains the point of symmetric quasicontinuity with respect to x_n , i.e. each interval of this hyperplane contains the point of continuity of this function. Thus, the function $f: \square^2 \rightarrow \square$ which is symmetrically quasicontinuous with respect to y has the following properties:

- i) is quasicontinuous with respect to x ,
- ii) Any interval parallel to the x -axis contains a point of continuity of the function f .

None of these properties in particular is sufficient for the function to be symmetrically quasincontinuous with respect to y . Hence here are the following examples:

Example 3.1. Let

$$f(x, y) = \begin{cases} 1, & \text{if } y \text{ is a rational number} \\ 0, & \text{if } y \text{ is an irrational number} \end{cases}.$$

A function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ that is defined in this way satisfies the property *i*), but it is not quasicontinuous at any point (x, y) , and therefore it is not symmetrically quasicontinuous at any point.

Example 3.2. We consider a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ which is defined as follows:

$$f(x, y) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q}, p, q \text{ mutually prime numbers} \end{cases}.$$

The function f is continuous at any point (x, y) , where x is a rational number. It also satisfies the property *ii*), but it is not symmetrically quasicontinuous with respect to y at the point (x, y) , where x is a rational number.

Note 3.1. Any function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ that satisfies both of the properties *i*) and *ii*) is symmetrically quasicontinuous with respect to y .

Proof. Let $\varepsilon > 0$ and $(x, y) \in \mathbb{R}^2$. Since the function f is quasicontinuous with respect to x , there is an interval $(a, b) \subset (x - \varepsilon, x + \varepsilon)$ such that for any $u \in (a, b)$ the following inequality holds

$$|f(u, y) - f(x, y)| < \frac{\varepsilon}{2}.$$

In the interval $(a, b) \times \{y\}$ there is a point of continuity of the function f .

The function f satisfies the property *ii*) which means that there is a $c \in (a, b)$ such that the function f is continuous at the point (c, y) . Hence, there exists a nonempty neighborhood U of the point (c, y) , such that for each point $(u, v) \in U$ we have

$$|f(u, v) - f(c, y)| < \frac{\varepsilon}{2}.$$

Using the previous inequalities, we obtain:

$$|f(u, v) - f(x, y)| \leq |f(u, v) - f(c, y)| + |f(c, y) - f(x, y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \blacklozenge$$

Since every continuous function is also quasicontinuous, we get a generalization of Note shown by V. Volterra (Volterra 1881), which means that every function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, that is continuous with respect to x and that has points of continuity in every interval parallel to x -axis, has a property that we here call quasicontinuity.

We will prove the following theorem:

Theorem 3.1. Any function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ that is continuous with respect to x and y is symmetrically quasicontinuous with respect to an arbitrary of the variables.

Proof. Let $\varepsilon > 0, \delta > 0$ and let $(x^0 - \delta, x^0 + \delta) \times (y^0 - \delta, y^0 + \delta)$ is a neighborhood of the point (x^0, y^0) .

Since the function f is continuous with respect to the variable x , then we can determine the positive number $\alpha < \delta$ that depends on ε and such that for $|x - x^0| < \alpha$ we have

$$|f(x, y^0) - f(x^0, y^0)| < \frac{\varepsilon}{2}. \quad (3.1)$$

We denote the set F_p of numbers x of the closed interval $[x^0 - \alpha, x^0 + \alpha]$ for which the inequality $|y - y^0| \leq \frac{\delta}{p}$ entails

$$|f(x, y) - f(x, y^0)| \leq \frac{\varepsilon}{2}. \quad (3.2)$$

Since the function f is continuous with respect to x , then the set F_p is closed.

However, the function under consideration is also continuous with respect to y , and therefore every point of the closed interval $[x^0 - \alpha, x^0 + \alpha]$ belongs to one of the sets F_p . At least one of those sets, for example, F_{p_0} , is not nowhere dense and as closed must contain the interval (a, b) . Let S be a rectangle $(a, b) \times \left(y^0 - \frac{\delta}{p_0}, y^0 + \frac{\delta}{p_0}\right)$. It is contained in the given neighborhood and even more it is symmetric with respect to the lines $y = y^0$.

Since for each point of the rectangle the inequalities (3.1) and (3.2) are satisfied, then

$$|f(x, y) - f(x^0, y^0)| \leq |f(x, y) - f(x, y^0)| + |f(x, y^0) - f(x^0, y^0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, the function f is continuous with respect to (x, y) and symmetric with respect to y in the point (x^0, y^0) . ♦

The following theorem, which is more general than the preceding one, will be proved by contradiction.

Theorem 3.2. If a function $f: \square^2 \rightarrow \square$ that is quasicontinuous with respect to x is continuous with respect to y , then f is quasicontinuous with respect to (x, y) and symmetric with respect to y .

Proof. It is sufficient to modify the proof of Theorem 2.1. ♦

Example 3.3. Let

$$f(x, y) = \begin{cases} 1 & \text{for } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{for the other points} \end{cases}.$$

The function f is quasicontinuous with respect to each of the variables, but it is not symmetrically quasicontinuous with respect to y at the points $(x, 0)$ for $x \geq 0$.

Therefore, in the Theorem 3.2 we cannot substitute continuity with respect to y with quasicontinuity.

Using the above theorem successively for variables and systems of variables, we conclude that a function $f: \square^n \rightarrow \square$ continuous with respect to each of the variables $x_1, x_2, \dots, x_n$ is quasicontinuous with respect to $(x_1, x_2, \dots, x_n)$ and symmetric with respect to x_i ($i = 1, 2, \dots, n$).

Since every function that is symmetrically quasicontinuous with respect to x_i ($i = 1, 2, \dots, n$) has points of continuity in every interval of the hyperplane $x_i = c$, then the continuous function with respect to each of the variables, is point discontinuous, since there are points of continuity in every interval of the hyperplane $x_i = c$.

This is a well-known conclusion as a final Note shown by M. Hahn (Hahn 1921).

QUASICONTINUOUS FUNCTION FROM UPPER AND LOWER

Definition 4.1. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *upper (lower) quasicontinuous* at a point $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ if for an arbitrarily $\varepsilon > 0$, in any neighborhood at that point there is an interval in which

$$f(x) < f(x^0) + \varepsilon \quad (f(x) > f(x^0) - \varepsilon).$$

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that is upper (lower) semicontinuous on one side, is also upper (lower) quasicontinuous.

Specifically, a monotonic function is everywhere upper and lower quasicontinuous. If the function has onesided limits at a certain point, then it is upper and lower quasicontinuous at that point.

From the definitions mentioned above, it follows that a quasicontinuous function must also be upper and lower quasicontinuous, but the converse theorem does not hold. For example, the function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\lim_{x_1 \rightarrow x_1^0-} f(x_1) < f(x_1^0) < \lim_{x_1 \rightarrow x_1^0+} f(x_1)$$

is not quasicontinuous at the point x_1^0 .

Theorem 4.1. If a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is everywhere upper quasicontinuous or everywhere lower quasicontinuous, then f is pointwise discontinuous.

Proof. Without losing generality, we can assume that a function f is bounded and everywhere upper quasicontinuous. Otherwise, we can consider a function $\arctan f$ that has the same set of points of continuity as the function f .

Let $V \subset \mathbb{R}^n$ be a non-empty open set. Let us show that in the set V there is a point of continuity of the function f . Let $m_1 = \inf_{x \in V} f(x)$. There is a point $x_1 \in V$ such as

$$f(x_1) - m_1 < 1.$$

Since the function f is upper quasicontinuous at the point x_1 , then there is a closed interval $P_1 \subset V$ such that for each point $x \in P_1$,

$$f(x) - f(x_1) < 1.$$

and we have $d(P_1) < 1$.

Let $m_2 = \inf_{x \in \text{Int } P_1} f(x)$. There is a point $x_2 \in \text{Int } P_1$ such as

$$f(x_2) - m_2 < \frac{1}{2}$$

and there is a closed interval $P_2 \subset \text{Int } P_1$ such that for each point $x \in P_2$,

$$f(x) - f(x_2) < \frac{1}{2} \quad \text{and} \quad d(P_2) < \frac{1}{2}.$$

In general, in k -th step we find the point $x_k \in \text{Int } P_{k-1}$ such that

$$f(x_k) - m_k < \frac{1}{k},$$

where $m_k = \inf_{x \in \text{Int } P_{k-1}} f(x)$ and there is a closed interval $P_k \subset \text{Int } P_{k-1}$ such that $d(P_k) < \frac{1}{k}$ and for each point $x \in P_k$,

$$f(x) - f(x_k) < \frac{1}{k}.$$

Let x_0 is a common point of the intervals P_k ($k=1,2,\dots$). It is obvious that $x_0 \in V$ and the function f is continuous at the point x_0 . Indeed, for any $\varepsilon > 0$, there is a number k such that $\frac{3}{k} < \varepsilon$. Then $x_0 \in \text{Int } P_k$ and

$$f(x_0) \geq m_k = \inf_{x \in \text{Int } P_{k-1}} f(x).$$

For each point $x \in P_k$,

$$f(x) - f(x_k) < \frac{1}{k} \quad \text{and} \quad f(x_k) < m_k + \frac{1}{k}.$$

As a result, for each point $x \in P_k$ the following inequality is applied:

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f(x_k)| + |f(x_k) - f(x_0)| < \\ &< |f(x) - f(x_k)| + |f(x_k) - m_k| < \frac{1}{k} + \frac{2}{k} = \frac{3}{k} < \varepsilon., \quad \blacklozenge \end{aligned}$$

Theorem 4.2. If a function $f: \square^n \rightarrow \square$ is pointwise discontinuous, then f is upper or lower quasicontinuous at each point.

Proof. We fix the point $x \in R^n$. Let U_k be a descending sequence of neighborhood of the point x for which:

$$\bigcap_{k=1}^{\infty} U_k = \{x\}.$$

Since the function f is pointwise discontinuous, then inside each neighborhood U_k there is a point x_k which is the point of continuity of the function f . As a result, for each $k=1,2,\dots$ exists a neighborhood V_k of the point x_k , such that the $x \in V_k$ inequality holds for each

$$|f(x) - f(x_k)| < \frac{1}{2k}.$$

Without losing generality, let assume that there exists $\lim_{k \rightarrow \infty} f(x_k)$ (finite or infinite).

Otherwise, we will restrict it to some subsequence.

There are the following options:

$$f(x) \geq \lim_{k \rightarrow \infty} f(x_k) \quad \text{or} \quad f(x) < \lim_{k \rightarrow \infty} f(x_k).$$

In the first case, the function f is upper quasicontinuous at the point x , and in the second case it is lower quasicontinuous at the point x . $\blacklozenge$

Repeating the procedure from the proof of Theorem 2.1 we can prove the following theorem:

Theorem 4.3. If the function $f: \square^2 \rightarrow \square$ is quasicontinuous with respect to x and upper quasicontinuous with respect to y , then f is upper quasicontinuous with respect to the point (x, y) .

Using the fact that a pointwise discontinuous function is upper or lower quasicontinuous at each point, and slightly varying the proof of Theorem 2.1, we obtain:

Theorem 4.4. If a function $f: \square^2 \rightarrow \square$ is pointwise discontinuous with respect to x and quasicontinuous with respect to y , then f is quasicontinuous at the point (x, y) at each point of continuity with respect to x .

Corollary 4.1. For a function of two variables to be pointwise discontinuous in a plane, it is sufficient to assume that it is quasicontinuous with respect to one of the variables and pointwise discontinuous with respect to the other.

Using the Theorem 4.4 for x_1 and $(x_2, x_3, \dots, x_n)$ and using Theorem 2.1, we note that a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ of n variables $x_1, x_2, \dots, x_n$ that is pointwise discontinuous with respect to x_1 and quasicontinuous with respect to each of the other variables, is pointwise discontinuous.

EQUALLY QUASICONTINUOUS FUNCTIONS

Definition 5.1. A function $f_s: \mathbb{R}^n \rightarrow \mathbb{R}$ that belongs to the family $\{f_s\}_{s \in S}$ (S is an index set) are called *equally quasicontinuous*, if for arbitrary $\varepsilon > 0$ and for an arbitrary neighborhood of the point x^0 , there is an interval that is contained in that neighborhood and in which the inequalities are fulfilled:

$$|f_s(x) - f_s(x^0)| < \varepsilon \quad \forall s \in S.$$

Therefore, a continuous function and a quasicontinuous function are equally quasicontinuous. It is obvious that the sum of two equally quasicontinuous functions is quasicontinuous.

When the same concept proof of Theorem 2.1 is repeated, it is obvious that:

Note 5.1. If a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ и $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ are equally quasicontinuous with respect to x and discontinuous with respect to y , then f and g are equally quasicontinuous with respect to the point (x, y) .

When a substitution is made in Note 5.1: x to x_1 and y to x_2 , then x to the point (x_1, x_2) and y to x_3 , and finally x to the point $(x_1, x_2, \dots, x_{n-1})$ and y to x_n , we have the following

Corollary 5.1. If functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous with respect to each of the variables, then f and g are equally quasicontinuous with respect to the point

$$(x_1, x_2, \dots, x_n).$$

Theorem 5.1. If functions $f_k: \mathbb{R}^n \rightarrow \mathbb{R}$ ($k=1, 2, \dots$) are equally quasicontinuous and $\lim_{k \rightarrow \infty} f_k = f$, then the function f is quasicontinuous.

Proof. We fix a point $x \in \mathbb{R}^n$ and a number $\varepsilon > 0$. Let U be a neighborhood of the point x .

Since the functions f_k are equally quasicontinuous at the point x , then there is an open interval $V \subset U$ such that for each number k and each point $t \in U$ the inequality holds

$$|f_k(t) - f_k(x)| < \frac{\varepsilon}{2}.$$

Also, for every k we have

$$|f_k(x) - f(x)| < \frac{\varepsilon}{2}.$$

As a result, for each number k and each point $t \in U$ stands:

$$\begin{aligned} |f(t) - f(x)| &\leq |f(t) - f_k(t)| + |f_k(t) - f_k(x)| + |f_k(x) - f(x)| < \\ &< |f(t) - f_k(t)| + \frac{\varepsilon}{2} + |f_k(x) - f(x)|. \end{aligned}$$

Letting $k \rightarrow \infty$ we have

$$|f(t) - f(x)| < \frac{\varepsilon}{2} < \varepsilon. \blacklozenge$$

Theorem 5.2. If functions $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ ($k=1,2,\dots$) are equally quasicontinuous and $\lim_{k \rightarrow \infty} f_k = f$, then the function f pointwise discontinuous.

Proof. It suffices to show that for any nonempty open set V and each number $\varepsilon > 0$, there is an open interval $U \subset V$ such that for each x and y that belong to U the inequality holds

$$|f(x) - f(y)| < \varepsilon.$$

Let V be an open set and let $\varepsilon > 0$. Since $\lim_{k \rightarrow \infty} f_k = f$, for each point $x \in V$ there is a natural number $k(x)$ such that for each $k \geq k(x)$ hold

$$|f_k(x) - f(x)| < \frac{\varepsilon}{6}.$$

Since $V = \bigcup_{k=1}^{\infty} V_k$, where $V_k = \{x \in V \mid k(x) = k\}$, then at least one of the sets V_k is nowhere dense. Let's choose this set to be V_{k_0} . Then, there are an open interval $U \subset V$ where the set V_{k_0} is dense in U . Hence, for each point $x \in U \cap V_{k_0}$ the inequality holds

$$|f_{k_0}(x) - f(x)| < \frac{\varepsilon}{6}.$$

Let $x \in \text{Int } U$. Because the function f_{k_0} is quasicontinuous in the point x , there is an open interval $Z \subset \text{Int } U$ such that for each $t \in Z$

$$|f_{k_0}(t) - f_{k_0}(x)| < \frac{\varepsilon}{6}.$$

Then, for each $u, v \in Z$:

$$|f_{k_0}(u) - f_{k_0}(v)| \leq |f_{k_0}(u) - f_{k_0}(x)| + |f_{k_0}(x) - f_{k_0}(v)| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

Let $m \geq k_0$. Then

$$|f_m(u) - f_m(v)| \leq \varepsilon.$$

Really, if $u, v \in V_{k_0}$ then

$$\begin{aligned} |f_m(u) - f_m(v)| &\leq |f_m(u) - f(u)| + |f(u) - f_{k_0}(u)| + |f_{k_0}(u) - f_{k_0}(v)| + \\ &+ |f_{k_0}(v) - f(v)| + |f(v) - f_m(v)| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon. \end{aligned}$$

If $u, v \notin V_{k_0}$, then, since the set V_{k_0} is dense in U and the function f_m is quasicontinuous (a quasicontinuous function is nearly continuous), there are sequences $\{u_k\}, \{v_k\}$ of both elements of the set $V_{k_0} \cap Z$ convergent respectively to u and v , and such that

$$f(u_k) \rightarrow f(u) \text{ and } f(v_k) \rightarrow f(v).$$

As a result, for $k=1,2,\dots$ we have

$$|f_m(u_k) - f_m(v_k)| < \varepsilon,$$

i.e. we have:

$$\lim_{k \rightarrow \infty} |f_m(u_k) - f_m(v_k)| = |f_m(u) - f_m(v)| < \varepsilon.$$

The proof will be completed if we put $m \rightarrow \infty$ in the above inequality ♦

Theorem 5.3. If the functions $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ ($k=1,2,\dots$) are quasicontinuous and the sequence $\{f_k\}$ is uniformly convergent to the function f , then f is a quasicontinuous function.

Proof. We fix the point $x \in \mathbb{R}^n$, number $\varepsilon > 0$ and the open neighborhood U of the point x .

From the uniform convergence of the sequence $\{f_k\}$ it follows that there is an index m such that

$$|f_m(x) - f(x)| < \frac{\varepsilon}{3} \text{ for each } x \in \mathbb{R}^n.$$

Since the function f_m is quasicontinuous at a point x , there exists a nonempty open set $V \subset U$ such that

$$|f_m(u) - f_m(x)| < \frac{\varepsilon}{3} \text{ for each } u \in V.$$

Therefore, for $u \in V$

$$|f(u) - f(x)| \leq |f(u) - f_m(u)| + |f_m(u) - f_m(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

i.e. the function f is quasicontinuous at a point x . ♦

Note 5.2. A composition of quasicontinuous functions does not have to be a quasicontinuous function.

Example 5.1. Consider the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x & \text{if } x \in \mathbb{R}^+ \\ -x & \text{if } x \in I^+ \\ -x+1 & \text{if } x \in \mathbb{R}^- \\ x-1 & \text{if } x \in I^- \end{cases} \text{ and } g(x) = \begin{cases} 0 & \text{if } x = 0 \\ x+1 & \text{if } x \in \mathbb{R}^+ \\ -x-1 & \text{if } x \in I^+ \\ -x & \text{if } x \in \mathbb{R}^- \\ x & \text{if } x \in I^- \end{cases}.$$

Note that composition $g \circ f$ is defined by:

$$(g \circ f)(x) = \begin{cases} 0 & \text{if } x = 0 \\ x+1 & \text{if } x \in \mathbb{R}^+ \\ -x & \text{if } x \in I^+ \\ -x+2 & \text{if } x \in \mathbb{R}^- \\ x-1 & \text{if } x \in I^- \end{cases}.$$

The function f is quasicontinuous at the point $x_0 = 0$, the function g is continuous at the point $f(x_0)$ and the composition $g \circ f$ is not quasicontinuous at any point.

Theorem 5.4. If the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is quasicontinuous and the function $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then the composition $g \circ f$ is a quasicontinuous function.

Proof. Let $x_0 \in \mathbb{R}$, $y_0 = f(x_0)$ and $z = g(y_0)$. We can show that the composition $g \circ f$ is quasicontinuous at the point x_0 .

Let V be the open neighborhood of the point z_0 and let U be the open neighborhood of the point x_0 . Since the function g is continuous, then $g^{-1}(V) = W$ it is the open neighborhood of the point y_0 . Since the function f is quasicontinuous at a point x_0 , then there exists a non-empty open set $Z \subset U$ such that $f(Z) \subset W$. Therefore, $g(f(Z)) \subset V$. ♦

Note 5.3. There is a quasicontinuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ and a continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that composition $f \circ g$ is not a quasicontinuous function at every point.

Example 5.2. Consider

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ x & \text{if } x \leq 0 \end{cases} \quad \text{and} \quad g(x) = x^2$$

for $x \in \mathbb{R}$.

Then

$$(f \circ g)(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

is not a quasicontinuous function at the point $x = 0$.

Note 5.4. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a homomorphism, then $f \circ g$ is a quasicontinuous function.

The proof is obvious.

CONCLUSION

The main goal of the presented paper, was to examine the basics of Kempisty theorems and to give a new and modern approach in proving these theorems. This purpose was established with hope that this approach will give easier way to use the underlying mathematical models in various applications. Models including quasicontinuity of many variables are widely used in many technical fields, such as Control systems (Estrada and Fridman 2008), (Polyakov and Chairez 2014) Thermodynamical systems and heat transfer (Kane and Northrop 2023), (Kick, Jantos and Junker 2022) as subjects of our scientific research. Furthermore, quasicontinuous functions have several applications in other technical fields such as: optimization problems, computational modelling, stress and strain analysis, vibrations etc.

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